Quaestiones Mathematicae 2022, 46(4): 775–779.

https://doi.org/10.2989/16073606.2022.2041759

NONMEASURABLE VITALI SET: VARIATIONS ON THEME

Angelo Bella
Department of Mathematics and Computer Sciences, University of Catania, Catania, 95125, Italy (EU).
E-Mail bella@dmi.unict.it

Daniele Puglisi*
Department of Mathematics and Computer Sciences, University of Catania, Catania, 95125, Italy (EU).
E-Mail dpuglisi@dmi.unict.it

Alfonso Villani
Department of Mathematics and Computer Sciences, University of Catania, Catania, 95125, Italy (EU).
E-Mail villani@dmi.unict.it

Abstract. We extend Vitali's procedure to get nonmeasurable sets. Answering a question of Kharazishvili [4], we present a relatively simple argument for passing from singletons to finite sets. Our result is actually much stronger, as it covers even the case of compact choices.

Mathematics Subject Classification (2020): Primary: 28C10; Secondary: 43A05.
Key words: Vitali set, Lebesgue measure, analitic and coanalitic sets.

1. A general Vitali selection. The set constructed by Vitali in 1905 [9] was the first example of a nonmeasurable subset of $\mathbb{R}$ with respect to the Lebesgue measure λ_1. The Vitali set is really ingenious, though simple in its own. Its construction strictly depends on the axiom of choice and indeed there is a model of ZF where all sets of real numbers are Lebesgue measurable [8].

Moreover, some Vitali subsets of the real line $\mathbb{R}$ can be measurable with respect to certain translation quasi-invariant measures on $\mathbb{R}$ extending the standard Lebesgue measure. On the other hand, for every nonzero σ-finite translation quasi-invariant measure on $\mathbb{R}$ there exist Vitali sets which are nonmeasurable with respect to it. Namely, assume G to be a group which acts on a set X, and H to be a subgroup. If μ denotes any G-invariant σ-additive measure on X, then an H-selector is a set having exactly one point in common with each orbit of H. Measurability of H-selector has been studied in [2] (see also [10]). In [7], S. Solecki investigates the

*Corresponding author.

existence of selectors which are nonmeasurable with respect to invariant extensions of μ. Moreover, in [6] it is shown that, under suitable assumption, each set of positive μ-measure contains a subset nonmeasurable with respect to any invariant extensions of μ. Recently, in [5] an extension on previous results has been obtained.

Our construction stands in the classical Vitali setting. For $a > 0$ let $X \subseteq [-a, a] \subseteq \mathbb{R}$ be any Lebesgue measurable subset such that $\lambda_1(X) > 0$. On X one considers the following equivalence relation:

$$x \sim y \quad \text{iff} \quad x - y \in \mathbb{Q}.$$

Let us denote by $X/\!\sim$ the quotient set and by $[x]$ the equivalence class containing the element x. Let $f : X/\!\sim \to \mathcal{P}(X) \setminus \{\emptyset\}$ be a function satisfying $f([x]) \subset [x]$ for every $[x] \in X/\!\sim$. f is nothing else than a choice function for the set $\{\mathcal{P}([x]) \setminus \{\emptyset\} : [x] \in X/\!\sim\}$. For each selection $f : X/\!\sim \to \mathcal{P}(X) \setminus \{\emptyset\}$ one defines

$$V^f = \bigcup_{[x] \in X/\sim} f([x])$$

Depending on the selections made by f, one can define the following family of sets. For each $n \in \mathbb{N}$,

$$\mathcal{V}_n = \left\{V^f : \ |f([x])| = n, \ \forall [x] \in X/\!\sim\right\};$$

$$\mathcal{V}_{\leq n} = \left\{V^f : \ |f([x])| \leq n, \ \forall [x] \in X/\!\sim\right\};$$

$$\mathcal{V}_{\text{fin}} = \left\{V^f : \ f([x]) \text{ is finite } \forall [x] \in X/\!\sim\right\}.$$

The classical Vitali theorem states that each set of $\mathcal{V}_1$ is nonmeasurable with respect to the Lebesgue measure λ_1.

In [4, Question 1 on p. 138] the author raises the question whether every set of $\mathcal{V}_{\leq n}$ is Lebesgue nonmeasurable. A proof is given there by using a profound theorem of S. Banach [1]. At the same time, the author asks if a simple argument for such a question can be given.

In this short note we give a simple proof that every element of $\mathcal{V}_{\leq n}$ is λ_1-nonmeasurable. It will be actually a consequence of a much stronger result.

Theorem 1. *Any extended Vitali set $V^f \in \mathcal{V}_{\text{fin}}$ is always λ_1-nonmeasurable.*

Proof. Let f be the finite set value mapping involved in the definition of V^f.

Assume by contradiction that V^f is λ_1-measurable. Let $V_1 \in \mathcal{V}_1$ be the Vitali set obtained by picking one point in each $f([x])$. Since V_1 is nonmeasurable and $V_1 \subset V^f$, it is clear that $\lambda_1(V^f) > 0$. Therefore, there should be a compact set

$$K \subseteq V^f \text{ such that } \lambda_1(K) > 0.$$

Let us define

$$A = \{x \in K : \ \exists y \in K \ x < y, \ x \sim y\}.$$

Since the map

$$(x, y) \in \mathbb{R}^2 \longmapsto y - x \in \mathbb{R}$$

is continuous, we have that

$$B = \{(x, y) \in \mathbb{R}^2 : \ y - x \in]0, +\infty[\cap \mathbb{Q}\}$$

is a Borel set, thus

$$B \cap (K \times K) := B_1 = \{(x, y) \in K \times K : \ x \sim y, \ x < y\}$$

is a Borel set in $K \times K$. Since

$$A = P_K(B_1),$$

where $P_K : K \times K \longrightarrow K$ the first component projection, we get that $K \setminus A$ is a coanalytic set and thus λ_1-measurable (see [3, Theorem 8.4.1]).

Let $[-a, 0] \cap \mathbb{Q} = \{r_p\}_{p \in \mathbb{N}}$ and define

$$K_p = (K \setminus A) + r_p.$$

Claim 1:

- $K_{p_1} \cap K_{p_2} = \emptyset$, for $p_1 \neq p_2$.

Indeed, if $x \in K_{p_1} \cap K_{p_2}$ then

$$x = y_1 + r_{p_1} = y_2 + r_{p_2},$$

with $y_1, y_2 \in K \setminus A$. One can suppose $y_1 < y_2$. But now we get in trouble, since $y_1 \sim y_2$ and then one should have $y_1 \in A$.

Claim 2:

- $K \subseteq \bigcup_{p \in \mathbb{N}} K_p$.

Indeed, let $x \in K$. Then either $x \in K \setminus A$ or $x \in A$. In the first case we are done. Otherwise, there must exist $y \in K$, with $x < y$ such that $y \in f([x])$. Since $K \subseteq V^f$ and $f([x])$ is finite, we can choose the maximum element $y \in f([x]) \cap K$; i.e. $y \in K \setminus A$. Thus

$$x = y + (x - y) \in (K \setminus A) + r_p \text{ for some } p \in \mathbb{N}.$$

Now,

$$0 < \lambda_1(K) \leq \lambda_1(\bigcup_{p \in \mathbb{N}} K_p) = \sum_{p \in \mathbb{N}} \lambda_1(K_p)$$

Since λ_1 is invariant by translation, we have $\lambda_1(K_p) = \lambda_1(K \setminus A)$ for every $p \in \mathbb{N}$. Therefore, from one hand the series $\sum_{p \in \mathbb{N}} \lambda_1(K_p)$ has to be $+\infty$; on the other hand

$$\bigcup_{p \in \mathbb{N}} K_p \subseteq [-2a, a],$$

and by monotonicity of λ_1, we have that the above series $\sum_{p\in\mathbb{N}} \lambda_1(K_p)$ has to be convergent. This contradiction concludes the proof that any $V^f \in \mathcal{V}_{\text{fin}}$ is not Lebesgue measurable. □

At first glance, it seems possible to have a similar result when the choices made by f are countable sets. But, being each equivalence class $[x]$ countable, it is evident that $V_{\text{countable}}$ can easily be measurable. Of course, one may wonder which choice functions guarantee that $V_{\text{countable}}$ is nonmeasurable. A strengthening of Theorem 1 shows this happens in a very interesting case.

Theorem 2. *The Vitali set obtained by choosing in every equivalence class a non-empty compact set is nonmeasurable.*

Proof. It suffices to argue as in the proof of Theorem 1. The key observation is that Claim 2 continues to hold because the set $f([x]) \cap K$, being a compact subset of $\mathbb{R}$, has maximum. □

Corollary. *The Vitali set obtained by choosing in every equivalence class a convergent sequence together with the limit point is nonmeasurable.*

Since any compact subset of an equivalence class is proper, the last theorem might suggest the question whether the Vitali set associated to a choice function f such that $f([x])$ is a non-empty proper subset of $[x]$ is always nonmeasurable. The following simple example proves this is not the case.

Example 1. Take $X = [-1, 1]$ and consider $f([x]) = [x] \cap [0, 1]$, for every $[x] \in X/\sim$. Therefore,

$$\emptyset \neq f([x]) \subsetneq [x], \text{ for every } [x] \in X/\sim$$

and

$$V^f = \bigcup_{[x]\in X/\sim} f([x])$$

is a measurable set. Indeed, $V^f = [0, 1]$.

Acknowledgement. The authors wish to thank David Fremlin for his suggestions. The first author was supported by Grant - Piano della Ricerca 2018-2020 - Projects “New trends in Combinatorics, Geometry and Topology”, University of Catania, Italy. The second and third authors were supported by Grant - Piano PIACERI 2020/22 linea di intervento 2, project “MAFANE”, University of Catania, Italy. The second author was also supported by Starting Grant Piano PIACERI 2020/22, linea di intervento 3, project “SOPSI”, University of Catania, Italy.

References

1. S. Banach, Sur le probleme de la mesure, *Fund. Math.* v. **4** (1923), 7–33.

2. J. Cichoń, A. Kharazishvili, and B. Weglorz, On sets of Vitali's type, *Proc. Amer. Math. Soc.* **118** (1993), 1243–1250.

3. D.L. Cohn, *Measure Theory,* second edition, Birkhäuser/Springer, New York, 2013.

4. A.B. Kharazishvili, *Set theoretical aspects of real analysis,* CRC Press, Boca Raton, FL, 2015.

5. J. Mirmina and D. Puglisi, Closed selections of Vitali's type, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **32** (2021), 751–755.

6. S. Solecki, On sets nonmeasurable with respect to invariant measures, *Proc. Amer. Math. Soc.* **119**(3) (1993), 115–124.

7. S. Solecki, Measurability properties of sets of Vitali's type, *Proc. Amer. Math. Soc.* **119**(3) (1993), 897–902.

8. R.M. Solovay, A model of set-theory in which every set of reals is Lebesgue measurable, *Annals of Math.* **92** (1970), 1–56.

9. G. Vitali, *Sul problema della misura dei gruppi di punti di una retta,* Tip. Gamberinie Parmeggiani, Bologna, 1905.

10. P. Zakrzewski, On nonmeasurable selectors of countable group actions, *Fund. Math.* **202**(3) (2009), 281–294.

Received 16 January, 2022.